5. SHELEPOV V.Iu.,On the index of an integral operator of the potential type in L_p space. Dokl. Akad. Nauk SSSR, Vol.186, No.6, 1969.
6. MUSKHELISHVILI N.I., Singular Integral Equations, Nauka, Moscow, 1968.
7. VAINBERG M.M. and TRENOGIN V.A., Theory of Bifurcation of Solutions of Non-linear Equations, Nauka, Moscow, 1969.
8. TRENOGIN V.A., Functional Analysis, Nauka, Moscow, 1980.
9. ESKIN G.I., Conjugate problem for equations of smooth type with two independent variables, Trudy Moskov. Matem. Obshch., Vol.21, 1970.

Translated by M.D.F.

PMM U.S.S.R.,Vol.47,No.6,pp.788-791,1983
Printed in Great Britain

0021-8928/83 $10.00+0.00

UDC 539.3

METHODS OF SOLVING SPATIAL PROBLEMS OF THE MECHANICS OF A DEFORMABLE SOLID IN TERMS OF STRESSES*

T. KHOLMATOV

The formulation in /1/ of a quasistatic problem of the mechanics of a deformable solid in terms of stresses is discussed, including also the variational formulation, which consists of solving six equations in six symmetric stress tensor components when six boundary conditions are satisfied. Methods of successive approximation are proposed for solving this problem and theorems on the convergence of these methods, including a "rapidly converging" method, whose rate of convergence is substantially higher than a geometric progression, are proved.

Utilization of the Lagrange and Castigliano variational principles in the numerical solution of boundary value problems of the mechanics of a deformable solid enables an a priori stable difference scheme /1/ to be compiled as well as an effective means for solving it. The disadvantages of applying each of these variational principles are well known. Thus, when using the Lagrangian, the desired quantities are displacements, and a numerical differentiation procedure that considerably reduces the accuracy of the solution obtained must be used to determine the state of stress. When using the Castiglianian, the problem is to seek the conditional extremum (in the class of tensor functions satisfying the equilibrium equations and the static boundary conditions), which often turns out to be difficult.

A new variational principle, based on solving the mechanics problem of a deformable solid in terms of stresses /1/ is considered below, and methods of solving the quasistatic problem of physically non-linear mechanics of a deformable solid are described.

1. Consider a physically non-linear medium in which the relation between the strain tensor components ε and the stress tensor components σ is given in the operator form

$$\varepsilon_{ij} = G_{ij}(\sigma) \tag{1.1}$$

On the boundary of a body Σ occupying a volume V let a force vector be given and let the following equilibrium conditions be satisfied:

$$\sigma_{ij}n_j|_\Sigma = S_i^\circ, \quad q_i|_\Sigma = -X_i|_\Sigma \tag{1.2}$$

(X_i are components of the volume force vector).

The quasistatic problem of the mechanics of a deformable body in terms of stresses (Problem B /1/) is to solve six equations in six unknown stress tensor components

$$E_{ijk,k} + Y_{ij} = 0 \tag{1.3}$$

while satisfying boundary conditions (1.2). Here

$$E_{ijk} \equiv \varepsilon_{ij,k} + \delta_{ki}(^1/_2\varepsilon_{mm,j} - \varepsilon_{mj,m}) + \delta_{kj}(^1/_2\varepsilon_{mm,i} - \varepsilon_{mi,m}) + \xi_{ij}(\varepsilon_{mk,m} - \varepsilon_{mm,k}) + R_i(q) + R_j(q) - \xi_{ij}R_k(q) \tag{1.4}$$

where their expressions in terms of the stresses (1.1) is substituted in place of the strain ε, ξ is a certain arbitrary symmetric constant tensor, and R is a certain linear vector

*Prikl.Matem.Mekhan.,47,6,988-992,1983

operator such that $R(q)=0$ only for $q=0$, while the symmetric tensor Y is given by the formula

$$Y_{ij}=R_{i,j}(X)+R_{i,j}(X)-\xi_{ij}R_{k,k}(X) \tag{1.5}$$

We assume that a scalar operator Ω dependent on the stress gradients exists and for which the conditions for the potentiality of the tensor (1.4) /2/ are satisfied: $E_{ijk}=\partial\Omega/\partial\sigma_{ij,k}$. Then the generalized solution of problem (1.2) and (1.3) can be found as the stationary point of the functional

$$I=\int_V(\Omega-Y_{ij}\sigma_{ij})\,dV-\int_\Sigma\chi_{ij}\sigma_{ij}\,d\Sigma+ \int_\Sigma\left[\frac{1}{2}(Aq_iq_i+B\sigma_{ij}n_j\sigma_{ik}n_k)+AX_iq_i-BS_i^\circ\sigma_{ij}n_j\right]d\Sigma \tag{1.6}$$

where A and B are certain dimensional constants different from zero, and the symmetric flux tensor $\chi_{ij}=E_{ijk}n_k$, defined on the surface Σ, does not vary.

The symmetric tensor σ satisfying the integral identity

$$\int_V E_{ijk}(\sigma)\tau_{ij,k}\,dV+f_\Sigma(\tau)=N(\tau)$$

$$N\equiv N^V+N_1^\Sigma+N_2^\Sigma,\quad N^V(\tau)\equiv\int_V Y_{ij}\tau_{ij}\,dV$$

$$N_1^\Sigma(\tau)\equiv\int_\Sigma\chi_{ij}(\sigma)\tau_{ij}\,d\Sigma,\quad N_2^\Sigma(\tau)\equiv\int_\Sigma(BS_i^\circ\tau_{ik}n_k-AX_i\tau_{ik,k})\,d\Sigma,$$

$$f_\Sigma(\tau)=\frac{1}{2}\int_\Sigma(Aq_i\tau_{ij,j}+B\sigma_{ij}n_j\tau_{ik}n_k)\,d\Sigma$$

for every smooth symmetric tensor τ /3/ is called the generalized solution of Problem B.

2. We consider a certain linear tensor-operator of the stress gradient

$$\pi_{ijk}=\Pi_{ijk}(\partial\sigma) \tag{2.1}$$

so that in a functional space $\partial\sigma\in D$ the quantity

$$(\sigma,\tau)_\pi=\int_V\pi_{ijk}(\partial\sigma)\tau_{ij,k}\,dV \tag{2.2}$$

satisfies all scalar product axioms /4/ such that the functional space D under consideration is a Hilbert space. Moreover, let the operator (2.1) be such that for an arbitrary symmetric tensor h_{ijk} in the first two subscripts, the following inequalities are satisfied

$$k\pi_{ijk}(h)h_{ijk}\leqslant\left[\frac{\partial E_{ijk}}{\partial\sigma_{lm,n}}h_{lmn}\right]h_{ijk}\leqslant K\pi_{ijk}(h)h_{ijk},\quad 0<k\leqslant K \tag{2.3}$$

If

$$\pi_{ijk}(\partial\sigma)=\frac{1}{2}(\delta_{il}\delta_{jm}+\delta_{im}\delta_{jn})\sigma_{lm,k}=\sigma_{ij,k} \tag{2.4}$$

then the first of the inequalities is equivalent to inequality (4.7) in /5/ for $k=k_0$.

When the operator Π in (2.4) is selected in this way, the Hilbert space D will be denoted by D_0.

Now, if a unique generalized solution of Problem B exists for the case when the operator of the governing relations (1.1) is the operator Π (2.1) (problem B_π), the method of successive approximations can be constructed

$$E_{ijk,k}\{\Pi(\partial\sigma^{(m+1)})\}=E_{ijk,k}\{\Pi(\partial\sigma^{(m)})\}-\beta^{(m)}[E_{ijk,k}\{G(\partial\sigma^{(m)})\}+Y_{ij}] \tag{2.5}$$

$$\sigma_{ij}^{(m+1)}n_j|_\Sigma=S_i^\circ,\quad q_i^{(m+1)}|_\Sigma=-X_i|_\Sigma$$

starting with a certain zeroth approximation $\sigma^{(0)}$ and setting $m=0,1,\ldots$

Theorem 1. Let a unique generalized solution exist for problem B_π, let conditions (2.3) hold, and let the given surface loads and volume forces satisfy the conditions

$$S^\circ\in L_p(\Sigma),\quad X|_\Sigma\in L_p(\Sigma),\quad p<{}^4/_3 \tag{2.6}$$

Moreover, let the following condition be satisfied for the zeroth approximation $\sigma^{(0)}$:

$$[E_{ijk}(\sigma^{(0)}) - \pi_{ijk}(\sigma^{(0)})]\, h_{ijk} \leqslant k\pi_{ijk}(h)\, h_{ijk} \tag{2.7}$$

where h is an arbitrary tensor of the third kind that is symmetric in the first two subscripts. Then in a certain neighbourhood

$$\| \sigma - \sigma^{(0)} \|_\pi \leqslant r \tag{2.8}$$

a generalized solution σ^* of Problem B exists that is unique in this neighbourhood, and for any value of the iteration parameter $\delta\ (0 < \beta < 2/K)$ the successive approximations process (2.5) reduces to it, starting with $\sigma^{(0)}$, where

$$\| \sigma^{(m)} - \sigma^* \|_\pi \leqslant \frac{q^m}{1-q} \| \sigma^{(1)} - \sigma^{(0)} \|_\pi \tag{2.9}$$

$$q \equiv \max(|1 - \beta k|,\ |1 - \beta K|) < 1$$

(The author formulated a special case of this theorem in /3/.)

To prove the theorem we examine the identity

$$\int\limits_V \pi_{ijk}(\sigma)\tau_{ij,k}\, dV = \int\limits_V \pi_{ijk}(\sigma)\tau_{ij,k}\, dV - \beta \left[\int\limits_V E_{ijk}(\sigma)\tau_{ij,k}\, dV + f_\Sigma(\tau) - N(\tau) \right] \tag{2.10}$$

On the left in (2.10) is the scalar product $(\sigma, \tau)_\pi$ in the space D_0. According to (2.3), the right side is a linear functional of τ in this space. Using the Sobolev imbedding theorem, it can be established that satisfaction of condition (2.6) is necessary for this. According to the Riesz theorem, this functional can be represented as the scalar product (σ', τ) where $\sigma' \in D_0$. Hence, a certain operator H sets each tensor function $\sigma \in D_0$ in correspondence with the tensor function $\sigma' \in D_0$. Thus, the question of finding the generalized solution of Problem B reduces to solving an operator equation of the following kind: $\sigma = H\sigma$.

From (2.10) and conditions (2.3) we have for the two tensor functions $\sigma^{(1)}$ and $\sigma^{(2)}$ and their difference $w = \sigma^{(2)} - \sigma^{(1)}$

$$|(H\sigma^{(2)} - H\sigma^{(1)}, w)_\pi| = |(w, w)_\pi - \beta \int\limits_V [E_{ijk}(\sigma^{(2)}) - E_{ijk}(\sigma^{(1)})]\, w_{ijk}\, dV| \leqslant q \| w \|_\pi^2 \tag{2.11}$$

$$w_{ijk} = \sigma^{(2)}_{lm,n} - \sigma^{(1)}_{lm,n}$$

where q is determined from the second relationship in (2.9). Here

$$|1 - \beta k| \geqslant |1 - \beta K|,\quad 0 < \beta \leqslant \beta_*$$
$$|1 - \beta K| \geqslant |1 - \beta k|,\quad \beta_* \leqslant \beta \leqslant 2/K$$
$$(\beta_* = 2/(k + K))$$

Hence, for $0 < \beta \leqslant 2/K$ the condition $q < 1$ is satisfied and inequality (2.11) is satisfied if

$$\| H\sigma^{(2)} - H\sigma^{(1)} \|_\pi \leqslant q \| \sigma^{(2)} - \sigma^{(1)} \|_\pi \tag{2.12}$$

Note that the least value $q = (K - k)/(K + m)$ of the quantity q is reached when $\beta = \beta_*$. Note also that the value of β can be changed at each iteration step so that $\beta^{(m)} \in (0, 2/K)$.

It follows from inequality (2.12) that the operator H realizes a compressed mapping /4/ in D_0.

Furthermore, we have

$$(H\sigma - H\sigma^{(0)}, \tau)_\pi = (H\sigma - H\sigma^{(0)}, \tau)_\pi + (H\sigma^{(0)} - \sigma^{(0)}, \tau)_\pi \tag{2.13}$$

But it follows from the identity (2.10) that

$$(H\sigma^{(0)} - \sigma^{(0)}, \tau)_\pi = \beta \int\limits_V [E_{ijk}(\sigma^{(0)}) - \pi_{ijk}(\sigma^{(0)})]\, \tau_{ij,k}\, dV \tag{2.14}$$

Applying condition (2.7) to (2.14) and setting $\tau = \sigma - \sigma^{(0)}$ in (2.13), we obtain

$$\| H\sigma - \sigma^{(0)} \| \leqslant (q + \beta K)\, r \leqslant r$$

i.e., the operator H that performs the compressed mapping, does not move any point from the circle (2.8). Hence, according to the compressed mapping principle, a generalized solution of Problem B exists. The uniqueness of this solution follows from /1/.

3. To obtain the convergence of the iteration process more rapidly than the convergence of a geometric progression, we impose a constraint on the second functional derivatives governing relationship (1.1). Let the inequality

$$\left|\left[\frac{\partial^2 E_{ijk}}{\partial\sigma_{lm,n}\sigma_{pq,r}} h_{lmn} h_{pqr}\right] h_{ijk}\right| \leqslant l\,(h_{ijk} h_{ijk})^{1/2}, \quad l > 0 \tag{3.1}$$

hold for an arbitrary third-rank tensor h that is symmetric in the first two subscripts.

Furthermore, we assume that the space D_1 with the scalar product

$$(\sigma, \tau)_1 = \int_V \left[\frac{\partial E_{ijk}}{\partial\sigma_{lm,n}} \sigma_{lmn}\right] \tau_{ij,k}\, dV \tag{3.2}$$

is Hilbertian for the tensor field σ, τ defined in the finite domain V. Then the following theorem holds.

Theorem 2 (The rapidly converging method /1/). Let a unique generalized solution of problem B_π exist, and let the inequalities (3.1) and

$$k h_{ijk} h_{ijk} \leqslant \left[\frac{\partial E_{ijk}}{\partial\sigma_{lm,n}} h_{lmn}\right] h_{ijk} \leqslant K h_{ijk} h_{ijk}, \quad 0 < k \leqslant K$$

be satisfied.

Moreover, let a be a positive number such that

$$\int_V [E_{ijk}(\sigma^{(0)}) - \overset{\circ}{E}_{ijk}(\sigma^{(0)})]\, \sigma^{(0)}_{ij,k}\, dV \leqslant n_1 a \int_V \sigma^{(0)}_{ij,k} \sigma^{(0)}_{ij,k}\, dV$$

Then there is a number α, $0 < \alpha \leqslant 1$ such that Problem B has a unique solution σ^* in the circle $\|\sigma^{(0)} - \sigma^*\|_1 \leqslant r_0$ if the inequality

$$q \leqslant a^{-\alpha} C; \quad q \equiv \frac{3}{2}\frac{l}{k} V^{-\alpha/2}, \quad C \equiv \alpha(1+\alpha)^{-(1+\alpha)/\alpha} \tag{3.3}$$

is satisfied, where r_0 is the least root of the equation $qr^{1+\alpha} - r + a = 0$.

When $\beta = 1$ the successive approximations beginning with $\sigma^{(0)}$ converge to this solution if we take as the operator $\overset{\circ}{E}_{ijk}$

$$\overset{\circ}{E}_{ijk}(h) = \frac{\partial E_{ijk}}{\partial\sigma_{lm,n}} h_{lmn}$$

where

$$\|\sigma^{(n)} - \sigma^*\|_1 \leqslant C_1 \delta^{(1+\alpha)^n}, \quad \delta = C^{1/\alpha}, \quad C_1 = \frac{a}{\delta(1-\delta)}$$

The proof follows from /1/.

REFERENCES

1. POBEDRYA B.E., Numerical Methods in the Theory of Elasticity and Plasticity, Izd-vo Moskovsk Gosudarst. Univ., Moscow, 1981.
2. POBEDRYA B.E., Lectures on Tensor Analysis, Izd-vo Mosk. Gos. Univ., Moscow, 1979.
3. KHOLMATOV T., On methods of solving problems in terms of stresses, Dokl. Akad. Nauk SSSR, Vol.252, No.2, 1980.
4. LYUSTERNIK L.A. and SOBOLEV V.I., Elements of Functional Analysis, Nauka, Moscow, 1965.
5. POBEDRYA B.E., Quasistatic problem of mechanics in stresses of a deformable solid, PMM, Vol.45, No.2, 1981.

Translated by M.D.F.